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Fair Optimization – Methodological Foundations of Fairness in Network Resource Allocation

(Invited Paper)

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***Abstract*—Network resource allocation problems are concerned with the allocation of limited resources among competing entities so as to respect some fairness rules while looking for the overall efficiency. This paper presents the methodology of fair optimization representing inequality averse optimization rather than strict inequality minimization as foundation of fairness in resource allocation. Commonly applied in network resource allocation Max-Min Fairness or the lexicographic maximin optimization are the most widely known concepts of fair optimization. Alternative models of fair optimization are discussed showing that they generate all the classical fair solution concepts as special cases. However, the fair optimization concepts can effectively generate various fair and efficient resource allocation schemes.**

***Keywords*—Fairness; resource allocation; equitability; fair optimization; lexicographic maximin**

I. INTRODUCTION

Resource allocation decisions are usually concerned with the allocation of limited resources so as to achieve the best system performance. However, in networking there is a need to respect some fairness rules while looking for the overall efficiency. A fair way of distribution of the bandwidth (or other network resources) among competing network entities (activities) becomes a key issue in computer networks and communication network design in general [1], [2]. In order to ensure fairness, all the entities have to be adequately provided with the resources. Nevertheless, fair treatment of all entities does not imply equal allocation of resources due to network constraints. This leads to concepts of fair optimization representing inequality averse optimization rather than strict inequality minimization. The so-called Max-Min Fairness (MMF) and its generalization to the lexicographic maximin optimization, which is widely applied in networking, is such a concept. As the MMF may cause a dramatic worsening of the overall efficiency [3], [4], several other fair allocation schemes are searched to get some tradeoff. We recall the concept of fair efficiency as a basis for fair optimization. It is a specific refinement of the Pareto-optimality which remains in harmony with both inequality minimization and outcomes maximization. Two alternative multicriteria models of fair optimization are discussed showing that they generate all the classical fairness solution concepts as special cases. Nevertheless, the fair optimization methodology provides a variety of techniques to generate fair and efficient solutions.

The paper is organized as follows. In the next section the fair optimization with the preference structure that complies with both the efficiency (Pareto-optimality) and with the Pigou-Dalton principle of transfers is used to formalize the fair solution concepts. In Section III the classical fairness solution concepts are presented as special cases of fair optimization. There is also shown that properties of convexity and positive homogeneity together with some boundedness condition are sufficient for a typical inequality measure to guarantee that it can be used consistently with the fair optimization rules. Further, two alternative multiple criteria models equivalent to fair optimization are introduced thus allowing to generate a larger variety of fair and efficient resource allocation schemes. In Section V we show how importance weights allocated to several entities can be introduced into fair optimization.

II. FAIR OPTIMIZATION

The generic resource allocation problem may be stated as follows. There is a system dealing with a set I of m entities (services, activities, agents). There is given a set Q of allocation patterns (allocation decisions). For each entity $i \in I$ a function $f_i(\mathbf{x})$ of the allocation pattern $\mathbf{x} \in Q$ is defined, which where the mean measures the outcome (effect) $y_i = f_i(\mathbf{x})$ of allocation pattern $\mathbf{x}$ for entity i. In network applications, a larger value of the outcome usually means a better effect (higher service quality). Otherwise, the outcomes can be replaced with their complements to some large number. Thus, we get a vector maximization problem:

$$\max \{\mathbf{f}(\mathbf{x}) \ : \ \mathbf{x} \in Q\} \tag{1}$$

where $\mathbf{f}(\mathbf{x})$ is a vector-function that maps the decision space $X = R^n$ into the criterion space $Y = R^m$, and $Q \subset X$ denotes the feasible set. In order to make model (1) operational, one needs to assume some solution concept specifying what it means to maximize multiple objective functions. The solution concepts may be defined by properties of the corresponding preference model. This is completely characterized by the relation of weak preference $\succeq$, while the corresponding relations of strict preference $\succ$ and indifference $\cong$ are defined by the following formulas: $\mathbf{y}' \succ \mathbf{y}'' \Leftrightarrow (\mathbf{y}' \succeq \mathbf{y}'' \ \text{ and } \ \mathbf{y}'' \not\succeq \mathbf{y}')$, $\mathbf{y}' \cong \mathbf{y}'' \Leftrightarrow (\mathbf{y}' \succeq \mathbf{y}'' \ \text{ and } \ \mathbf{y}'' \succeq \mathbf{y}')$. The standard preference model related to the Pareto-optimal (efficient) solution concept assumes that the preference relation $\succeq$ is *reflexive*:

$$\mathbf{y} \succeq \mathbf{y}, \tag{2}$$

transitive:

$$(\mathbf{y}' \succeq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \succeq \mathbf{y}''') \quad \Rightarrow \quad \mathbf{y}' \succeq \mathbf{y}''', \tag{3}$$

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and *strictly monotonic*:

$$\mathbf{y} + \varepsilon \mathbf{e}_i \succ \mathbf{y} \quad \text{for } \varepsilon > 0;\ i = 1, \dots, m, \tag{4}$$

where $\mathbf{e}_i$ denotes the i–th unit vector in the criterion space. The last assumption expresses that for each individual objective function more is better (maximization). The preference relations satisfying axioms (2)–(4) are referred to as *rational preference relations*. Outcome vector $\mathbf{y}'$ rationally dominates $\mathbf{y}''$ ($\mathbf{y}' \succ_r \mathbf{y}''$), iff $\mathbf{y}' \succ \mathbf{y}''$ for all rational preference relations $\succeq$. The dominance relation $\mathbf{y}' \succeq_r \mathbf{y}''$ may be expressed in terms of the vector inequality $y_i' \geq y_i''$ for all $i \in I$. A feasible solution $\mathbf{x} \in Q$ is called *Pareto-optimal (efficient)* solution of the multiple criteria problem (1), iff $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is rationally nondominated, i.e., one cannot improve any outcome without worsening another.

In order to ensure fairness, all entities have to be equally well provided with the system's resources. This leads to concepts of fairness expressed by the fair (equitable) rational preferences [5]. First of all, the fairness requires impartiality of evaluation, thus focusing on the distribution of outcome values while ignoring their ordering. Hence, the preference model is impartial (anonymous, symmetric), i.e.,

$$(y_{\pi(1)}, \dots, y_{\pi(m)}) \cong (y_1, \dots, y_m) \quad \forall\ \pi \in \Pi(I) \tag{5}$$

where $\Pi(I)$ denotes the set of all permutations of I. This means that any permuted outcome vector is indifferent in terms of the preference relation. Further, fairness requires equitability of outcomes which causes that the preference model should satisfy the (Pigou–Dalton) principle of transfers. The principle of transfers states that a transfer of any small amount from an outcome to any other relatively worse-off outcome results in a more preferred outcome vector, i.e., whenever $y_{i'} > y_{i''}$ then

$$\mathbf{y} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''} \succ \mathbf{y} \quad \text{for } 0 < \varepsilon < (y_{i'} - y_{i''}) \tag{6}$$

The rational preference relations satisfying additionally properties (5) and (6) are called *fair (equitable) rational*. Outcome vector $\mathbf{y}'$ *fairly dominates* $\mathbf{y}''$, ($\mathbf{y}' \succ_e \mathbf{y}''$), iff $\mathbf{y}'$ is preferred to $\mathbf{y}''$ for all fair rational preference relations. In other words, $\mathbf{y}'$ fairly dominates $\mathbf{y}''$, if there exists a finite sequence of vectors $\mathbf{y}^j$ ($j = 1, 2, \dots, s$) such that $\mathbf{y}^1 = \mathbf{y}''$, $\mathbf{y}^s = \mathbf{y}'$ and $\mathbf{y}^j$ is constructed from $\mathbf{y}^{j-1}$ by application of either permutation of coordinates, equitable transfer, or increase of a coordinate. An allocation pattern $\mathbf{x} \in Q$ is called *fairly optimal* or *fairly efficient* if $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is fairly nondominated. Every fairly optimal solution is also Pareto-optimal, but not vice verse. Fair optimization depends on finding fairly optimal solutions. Specific fair solution concepts are defined by optimization according to a fairly rational preference relation (note that the relation definition is different from that in [6]).

III. Classical Fair Optimization Concepts

Specific fair solution concepts are defined by optimization according to a fairly rational preference relation (note that the relation definition is, in general, different from that considered in [6]). Simple solution concepts are based on maximization of some aggregation (or utility) functions $g : Y \to R$:

$$\max\ \{g(\mathbf{f}(\mathbf{x}))\ :\ \mathbf{x} \in Q\}. \tag{7}$$

i.e., by preference relation $\mathbf{y}' \succeq \mathbf{y}''$ iff $g(\mathbf{y}') \geq g(\mathbf{y}'')$. In order to guarantee the consistency of the aggregated problem (7) with the maximization of all individual objective functions (or Pareto-optimality of the solution), the aggregation function must be strictly increasing with respect to every coordinate. Following the requirements of impartiality (5) and the principle of transfers (6), to guarantee fairness of the solution concept (7), the aggregation function must also be symmetric, i.e. for any permutation π of I, $g(y_{\pi(1)}, y_{\pi(2)}, \dots, y_{\pi(m)}) = g(y_1, y_2, \dots, y_m)$ as well as be equitable in the sense that $g(y_1, \dots, y_{i'} - \varepsilon, \dots, y_{i''} + \varepsilon, \dots, y_m) > g(y_1, y_2, \dots, y_m)$ for any $0 < \varepsilon < y_{i'} - y_{i''}$. Such functions are referred to as (strictly) Schur-concave [7]. In the case of a strictly increasing and strictly Schur-concave function, every optimal solution to the aggregated problem (7) defines some fairly optimal solution of problem (1) [5].

The simplest aggregation functions commonly used for the multiple criteria problem (1) are defined as the total outcome (the total throughput in typical network problems) $T(\mathbf{y}) = \sum_{i \in I} y_i$, equivalently as the mean (average) outcome $\mu(\mathbf{y}) = T(\mathbf{y})/m$ or alternatively as the worst outcome $M(\mathbf{y}) = \min_{i \in I} y_i$. The mean (total) outcome maximization is primarily concerned with the overall efficiency. It may generate solutions where some entities are discriminated in terms of performances even leading to starvation of many processes. Maximization of the worst outcome $M(\mathbf{y})$, i.e., the so-called maximin solution concept is regarded as maintaining equity. Indeed, in the case if the perfect equity solution is feasible and Pareto-optimal, then it is the unique optimal solution of the maximin model [8]. In general, the maximin model does not guarantee equity either efficiency. The maximin solution may be, however, regularized according to the Rawlsian principle of justice. Formalization of this concept leads us to the lexicographic maximin (LMM) optimization model where the largest feasible performance function value for activities with the smallest (i.e., worst) performance function value (this is the maximin solution), is followed by the largest feasible performance function value for activities with the second smallest (i.e., second worst) performance function value, without decreasing the smallest value, and so forth. The seminal book [9] brings together much of the LMM based so-called equitable resource allocation research from the past thirty years and provides current state of art in models and algorithm within wide gamut of applications. Within the communications or network applications the LMM approach has appeared already in [10], [11] as the MMF solution concept defined by the lack of a possibility to increase of any outcome without decreasing of some smaller outcome [11] and now it is treated as one of the standard fairness concepts [12]. In the case of convex attainable set (as considered in [11]) such a characterization represents also the LMM solution. In nonconvex case, as pointed out in [13], such strictly defined MMF solution may not exist while the LMM always exists and it covers the former if exists (see [14] for wider discussion). Therefore, the MMF is commonly identified with the LMM while the classical MMF definition is considered rather as an algorithmic approach which is applicable only for convex models. Indeed, while for convex problems it is relatively easy to form sequential algorithms to execute LMM by recursive maximin optimization with fixed smallest outcomes (see [9], [12], [14]–[16]), for nonconvex problems the sequential algorithms must be built with the use of some artificial criteria (see [8], [15], [17], [18] and [9, Ch. 7]).

For any strictly concave and strictly increasing utility function $u : R \rightarrow R$, the mean utility aggregation $g(\mathbf{y}) = \mu(u(\mathbf{y})) = \sum_{i\in I} u(y_i)/m$ is a strictly monotonic and strictly Schur-concave function thus defining a family of the fair aggregations [5]. Various concave utility functions u can be used to define fair solution concepts. In the case of positive outcomes, like in most network resource allocation problems, one may use the logarithmic function thus resulting in the *Proportional Fairness* (PF) solution concept [19]. Actually, it corresponds to the so-called Nash criterion which maximizes the product of additional utilities compared to the status quo. For positive outcomes also a parametric class of utility functions:

$$u(y_i, \alpha) = \begin{cases} y_i^{1-\alpha}/(1-\alpha) & \text{if } \alpha \neq 1 \\ \log(y_i) & \text{if } \alpha = 1 \end{cases}$$

may be used to generate various fair solution concepts for $\alpha > 0$ [20]. The corresponding solution concept, called α-fairness, represents the PF approach for $\alpha = 1$, while with α tending to the infinity it converges to the LMM. For large enough α one gets generally an approximation to the LMM while for discrete problems large enough α guarantee the exact LMM solution. Such a way to identify the LMM solution was considered in location problems [17] as well as to content distribution networking problems [21]. For a common case of upper bounded outcomes $y_i \leq u^*$ one may maximize power functions $-\sum_{i=1}^m (u^* - y_i)^p$ for $1 < p < \infty$ which is equivalent to minimization of the corresponding p-norm distances from the common upper bound u^* [5].

In system analysis fairness is usually quantified with so-called fairness measures (or inequality measures), which are functions ϱ that maps $\mathbf{y}$ into (nonnegative) real numbers. Various measures have been proposed throughout the years, e.g., in [22]–[27] and references therein. Typical inequality measures are deviation type dispersion characteristics. They are *translation invariant* in the sense that $\varrho(\mathbf{y} + a\mathbf{e}) = \varrho(\mathbf{y})$ for any real number a (where $\mathbf{e}$ vector of units $(1, \ldots, 1)$), thus being not affected by any shift of the outcome scale. Moreover, the inequality measures are also *inequality relevant* which means that they are equal to 0 in the case of perfectly equal outcomes while taking positive values for unequal ones, thus to be minimized for fairness. Although some fairness measures, like Jain's index requires maximization.

The simplest inequality measures are based on the absolute measurement of the spread of outcomes, like the *maximum absolute difference* or the *mean absolute difference*

$$\Gamma(\mathbf{y}) = \tfrac{1}{2m^2} \textstyle\sum_{i\in I} \sum_{j\in I} |y_i - y_j|. \tag{8}$$

Another group of measures is related to deviations from the mean outcome, like the *maximum absolute deviation* or the *mean absolute deviation*

$$\delta(\mathbf{y}) = \tfrac{1}{m} \textstyle\sum_{i\in I} |y_i - \mu(\mathbf{y})|. \tag{9}$$

The *standard deviation* σ (or the *variance* σ^2) represents both the deviations and the spread measurement as

$$\sigma^2(\mathbf{y}) = \frac{1}{m} \sum_{i\in I} (y_i - \mu(\mathbf{y}))^2 = \frac{1}{2m^2} \sum_{i\in I} \sum_{j\in I} (y_i - y_j)^2. \tag{10}$$

Deviational measures may be focused on the downside semideviations as related to worsening of outcome while ignoring upper semideviations related to improvement of outcome. One may define the *maximum (downside) semideviation*

$$\Delta(\mathbf{y}) = \max_{i\in I} (\mu(\mathbf{y}) - y_i) = \mu(\mathbf{y}) - M(\mathbf{y}) \tag{11}$$

and the *mean (downside) semideviation*

$$\bar{\delta}(\mathbf{y}) = \tfrac{1}{m} \textstyle\sum_{i\in I} (\mu(\mathbf{y}) - y_i)_+ \tag{12}$$

where $(.)_+$ denotes the nonnegative part of a number. Similarly, the *standard (downside) semideviation* is given as

$$\bar{\sigma}(\mathbf{y}) = \sqrt{\tfrac{1}{m} \textstyle\sum_{i\in I} (\mu(\mathbf{y}) - y_i)_+^2}. \tag{13}$$

In economics there are usually used relative inequality measures normalized by mean outcome, so-called indices. The most commonly accepted is the Gini index (Gini coefficient) $G(\mathbf{y}) = \Gamma(\mathbf{y})/\mu(\mathbf{y})$, which is the relative mean difference. Considered in networking the Jain's index [23] computes a normalized square mean as $J(\mathbf{y}) = 1 - \sigma^2(\mathbf{y})/\mu(\mathbf{y}^2)$. One can easily notice that direct minimization of typical inequality measures (especially the relative ones) may contradict the optimization of individual outcomes resulting in equal but very low outcomes. The same applies to the Jain's index maximization. Moreover, this contradiction cannot completely be resolved with the standard bicriteria mean-equity model [24] which takes into account both the efficiency with optimization of the mean outcome $\mu(\mathbf{y})$ and the equity with minimization of an inequality measure $\varrho(\mathbf{y})$.

Note that the lack of consistency of the mean-equity model with the outcomes maximization applies also to the case of the maximum semideviation $\Delta(\mathbf{y})$ (11) used as an inequality measure whereas subtracting this measure from the mean $\mu(\mathbf{y}) - \Delta(\mathbf{y}) = M(\mathbf{y})$ results in the worst outcome and thereby the first criterion of the LMM model. In other words, although a direct use of the maximum semideviation in the mean-equity model may contradict the outcome maximization, the measure can be used complementary to the mean leading us to the worst outcome criterion which does not contradict the outcome maximization. This construction can be generalized for various (dispersion type) inequality measures. Moreover, we allow the measures to be scaled with any positive factor $\alpha > 0$. For any inequality measure ϱ we introduce the corresponding underachievement function defined as the difference of the mean outcome and the (scaled) inequality measure itself, i.e.

$$M_{\alpha\varrho}(\mathbf{y}) = \mu(\mathbf{y}) - \alpha\varrho(\mathbf{y}). \tag{14}$$

We say that (dispersion type) inequality measure $\varrho(\mathbf{y}) \geq 0$ is *strictly* Δ*-bounded* if it is upper bounded by the maximum downside deviation $\varrho(\mathbf{y}) \leq \Delta(\mathbf{y})\ \forall \mathbf{y}$ and the inequality is strict except from the case of perfectly equal outcomes, i.e., $\varrho(\mathbf{y}) < \Delta(\mathbf{y})$ for any $\mathbf{y}$ such that $\Delta(\mathbf{y}) > 0$. If $\alpha_0\varrho(\mathbf{y})$ is strictly Δ-bounded, then a positively homogeneous and translation invariant (dispersion type) inequality measure $\varrho(\mathbf{y}) \geq 0$ generates the monotonic underachievement function $M_{\alpha\varrho}(\mathbf{y})$ for any $0 < \alpha \leq \alpha_0$ [8]. Hence, any such a strictly Schur-convex inequality measure ϱ defines a fair solution concept. This applies, in particular, to the mean absolute difference (8) generating a proper fair solution concept

$$M_{\alpha\Gamma}(\mathbf{y}) = \frac{1-\alpha}{m} \sum_{i\in I} y_i + \frac{\alpha}{m^2} \sum_{i\in I} \sum_{j\in I} \min\{y_i, y_j\} \tag{15}$$

for any $0 < \alpha \leq 1$. Similar result is valid for the standard semideviation (13) but not for variance [8].

IV. Multicriteria Models of Fair Optimization

The relation of fair dominance can be expressed as a vector inequality on the cumulative ordered outcomes [28]. The latter can be formalized as follows. First, we introduce the ordering map $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \ldots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \leq \theta_2(\mathbf{y}) \leq \cdots \leq \theta_m(\mathbf{y})$ and there exists a permutation π of set I such that $\theta_i(\mathbf{y}) = y_{\pi(i)}$ for $i \in I$. Next, we apply cumulation to the ordered outcome vectors to get quantities

$$\bar{\theta}_i(\mathbf{y}) = \sum_{j=1}^{i} \theta_j(\mathbf{y}) \quad \text{for } i \in I \tag{16}$$

expressing, respectively, the worst outcome, the total of the two worst outcomes, the total of the three worst outcomes, etc. Pointwise comparison of the cumulative ordered outcomes $\bar{\Theta}(\mathbf{y})$ for vectors with equal means was studied within the theory of equity [29] or the mathematical theory of majorization [7], where it is called the relation of Lorenz dominance or weak majorization, respectively. It includes the classical results allowing to express an improvement in terms of the Lorenz dominance as a finite sequence of equitable transfers (6). It can be generalized to vectors with various means [28] justifying that outcome vector $\mathbf{y}'$ fairly dominates $\mathbf{y}''$, iff $\bar{\theta}_i(\mathbf{y}') \geq \bar{\theta}_i(\mathbf{y}'')$ for all $i \in I$ where at least one strict inequality holds. Hence, fairly optimal solutions to problem (1) can be generated as Pareto-optimal solutions for the multiple criteria problem

$$\max \{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\}. \tag{17}$$

Note, that the aggregation maximizing the total outcome, corresponds to maximization of the last objective $\bar{\theta}_m(\mathbf{f}(\mathbf{x}))$ in problem (17). Similar, the maximin corresponds to maximization of the first objective $\bar{\theta}_1(\mathbf{f}(\mathbf{x}))$. As limited to a single criterion they do not guarantee the fairness of the optimal solution. On the other hand, when applying the lexicographic optimization to problem (17)

$$\operatorname{lexmax} \{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\} \tag{18}$$

one gets the lexicographic maximin solution concept,

$$\operatorname{lexmax} \{(\theta_1(\mathbf{f}(\mathbf{x})), \theta_2(\mathbf{f}(\mathbf{x})), \ldots, \theta_m(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\} \tag{19}$$

i.e., the classical equitable optimization [9] representing LMM.

For modeling various fair preferences one may use some combinations of the criteria in problem (17). In particular, for the weighted sum aggregation on gets $\sum_{i \in I} s_i \bar{\theta}_i(\mathbf{y})$, which can be expressed in the form with weights $\omega_i = \sum_{j=i}^{m} s_j$ $(i = 1, \ldots, m)$ allocated to the ordered outcomes, i.e., as the so-called Ordered Weighted Average (OWA) [30]:

$$\max \{\textstyle\sum_{i \in I} \omega_i \theta_i(\mathbf{f}(\mathbf{x})) \ : \ \mathbf{x} \in Q\}. \tag{20}$$

If weights ω_i are strictly decreasing and positive, i.e. $\omega_1 > \omega_2 > \cdots > \omega_m > 0$, then each optimal solution of the OWA problem (20) is fairly optimal. Such OWA aggregations are sometimes called Ordered Ordered Weighted Averages [31]. Fair solution concept (15) based on the mean absolute difference is actually such an OWA with constantly decreasing weights $\omega_i - \omega_{i+1} = 2\lambda/m^2$ [24]. When differences between weights tend to infinity, the OWA model becomes LMM [32].

The definition of quantities $\bar{\theta}_k(\mathbf{y})$ is complicated as requiring ordering. Nevertheless, the quantities themselves can be modeled with simple auxiliary variables and linear constraints. Although, maximization of the k-th smallest outcome is a hard (combinatorial) problem. The maximization of the sum of k smallest outcomes is an LP problem as $\bar{\theta}_k(\mathbf{y}) = \max_t \ (kt - \sum_{i \in I}(t - y_i)_+)$ where t is an unrestricted variable. This allows one to implement the OWA optimization quite effectively as an extension of the original constraints and criteria with simple linear inequalities [33] and solve various network resource allocation problems [34], [35] as well as to define sequential methods for lexicographic maximin optimization of discrete and non-convex models [18], [36]. Various fairly optimal solutions of (1) may be generated as Pareto-optimal solutions to multicriteria problem (17).

The ordered outcome vectors describe a distribution of outcomes generated by a given allocation $\mathbf{x}$. In the case when there exists a finite set of all possible outcomes of the individual objective functions, we can directly deal with the distribution of outcomes described by frequencies of several outcomes. However, in order to take into account the principle of transfers we need to distinguish values of outcomes smaller or equal to the target value thus focusing on mean shortfalls (mean below-target deviations) to outcome targets τ:

$$\bar{\delta}_\tau(\mathbf{y}) = \tfrac{1}{m} \textstyle\sum_{i \in I}(\tau - y_i)_+. \tag{21}$$

It turns out that one may completely characterize the fair dominance by the pointwise comparison of the mean shortfalls for all possible targets. Outcome vector $\mathbf{y}'$ fairly dominates $\mathbf{y}''$, iff $\bar{\delta}_\tau(\mathbf{y}') \leq \bar{\delta}_\tau(\mathbf{y}'')$ for all $\tau \in R$ where at least one strict inequality holds [8].

For m-dimensional outcome vectors we consider, all the shortfall values are completely defined by the shortfalls for at most m different targets representing values of several outcomes y_i while the remaining shortfall values follow from the linear interpolation. Nevertheless, these target values are dependent on specific outcome vectors and one cannot define any universal grid of targets allowing to compare all possible outcome vectors. In order to take advantages of the multiple criteria methodology one needs to focus on a finite set of target values. Let $\tau_1 < \tau_2 < \cdots < \tau_r$ denote the all attainable outcomes. Fair solutions to problem (1) can be expressed as Pareto-optimal solutions for the multiple criteria problem with objectives $\bar{\delta}_{\tau_j}(\mathbf{f}(\mathbf{x}))$:

$$\min \{(\bar{\delta}_{\tau_1}(\mathbf{f}(\mathbf{x})), \bar{\delta}_{\tau_2}(\mathbf{f}(\mathbf{x})), \ldots, \bar{\delta}_{\tau_r}(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\}. \tag{22}$$

Hence, the multiple criteria problem (22) may serve as a source of fair solution concepts. When applying the lexicographic minimization to problem (22) one gets the lexicographic maximin solution concept, i.e., the classical equitable optimization model [9] representing the LMM. However, for the lexicographic maximin solution concept one simply perform lexicographic minimization of functions counting outcomes not exceeding several targets [17], [18], [36]. Certainly in many network resource allocation problems one cannot consider target values covering all attainable outcomes. In order to get a computational procedure one needs to focus on arbitrarily preselected finite grid of targets. By reducing the number of targets one restricts opportunities to generate all possible fair allocations. Nevertheless, one may still generate reasonable compromise solutions [8], [37].

V. Fair Optimization with Importance Weights

Frequently, one may be interested in putting into allocation models some additional entity weights $v_i > 0$. Typically the model of distribution weights is introduced thus defining distribution of outcomes $y_i = f_i(\mathbf{x})$ according to measures defined by the weights v_i for $i = 1, \dots, m$. We will use the normalized weights $\bar{v}_i = v_i / \sum_{i \in I} v_i$, rather than the original quantities v_i. Note that, in the case of unweighted problem (all $v_i = 1$), all the normalized weights are given as $\bar{v}_i = 1/m$. The importance weights can be easily accommodated in solution concept of the mean outcome $\mu(\mathbf{y}) = \sum_{i \in I} \bar{v}_i y_i$ as well as in most typical inequality measures and thereby in the corresponding underachievement measures (14). In particular, in the mean absolute difference based underachievement measure (15) as

$$M_{\alpha\Gamma}(\mathbf{y}) = (1-\alpha) \sum_{i \in I} \bar{v}_i y_i + \alpha \sum_{i \in I} \sum_{j \in I} \bar{v}_i \bar{v}_j \min\{y_i, y_j\}.$$

Similarly, for any utility function $u : R \rightarrow R$ one gets $\mu(u(\mathbf{y})) = \sum_{i \in I} \bar{v}_i u(y_i)$.

The fair dominance for general weighted problems can be derived by their disaggregation to the unweighted ones [38]. It can be mathematically formalized as follows. First, we introduce the right-continuous cumulative distribution function (cdf):

$$F_{\mathbf{y}}(d) = \sum_{i \in I} \bar{v}_i \delta_i(d), \quad \delta_i(d) = \begin{cases} 1 & \text{if } y_i \le d \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

which for any real (outcome) value d provides the measure of outcomes smaller or equal to d. Next, we introduce the quantile function $F_{\mathbf{y}}^{(-1)}$ as the left-continuous inverse of the cumulative distribution function $F_{\mathbf{y}}$:

$$F_{\mathbf{y}}^{(-1)}(\beta) = \inf\ \{\eta : F_{\mathbf{y}}(\eta) \ge \beta\} \quad \text{for } 0 < \beta \le 1.$$

By integrating $F_{\mathbf{y}}^{(-1)}$ one gets $F_{\mathbf{y}}^{(-2)}(0) = 0$ and

$$F_{\mathbf{y}}^{(-2)}(\beta) = \int_0^\beta F_{\mathbf{y}}^{(-1)}(\alpha) d\alpha \quad \forall\ 0 < \beta \le 1, \tag{24}$$

where $F_{\mathbf{y}}^{(-2)}(1) = \mu(\mathbf{y})$. The graph of function $F_{\mathbf{y}}^{(-2)}(\beta)$ (with respect to β) take the form of concave curves. It is called Absolute Lorenz Curve (ALC) [39], due to its relation to the classical Lorenz curve used in income economics as a cumulative population versus income curve to compare equity of income distributions. The ALC defines the relation (partial order) equivalent to the fair dominance. Exactly, outcome vector $\mathbf{y}'$ fairly dominates $\mathbf{y}''$, iff $F_{\mathbf{y}'}^{(-2)}(\beta) \ge F_{\mathbf{y}''}^{(-2)}(\beta)$ for all $\beta \in (0, 1]$ where at least one strict inequality holds. Note that for the case of unweighted outcomes, the ALC is completely defined by the values of the (cumulated) ordered outcomes. Hence, $\bar{\theta}_i(\mathbf{y}) = mF_{\mathbf{y}}^{(-2)}(i/m)$ for $i = 1, \dots, m$, and pointwise comparison of cumulated ordered outcomes is enough to justify fair dominance. In general case more β levels must be considered. Although, similarly to the cumulated ordered outcomes, maximization of a quantity $F_{\mathbf{y}}^{(-2)}(\beta)$ is an LP problem as $F_{\mathbf{y}}^{(-2)}(\beta) = \max_t\ (t - \frac{1}{\beta} \sum_{i \in I} \bar{v}_i (t - y_i)_+)$ where t is an unrestricted variable.

Within the weighted model, impartiality of the allocation process (5) is considered in terms that two allocation schemes leading to the same distribution of outcomes are indifferent

$$F_{\mathbf{y}'} = F_{\mathbf{y}''} \quad \Rightarrow \quad \mathbf{y}' \cong \mathbf{y}''. \tag{25}$$

The principle of transfers (6) is considered for single units of service. Although it can can be applied directly to the outcomes of importance weighted entities in the following form: if $y_{i'} > y_{i''}$ then

$$\mathbf{y}^\varepsilon = \mathbf{y} - \tfrac{\varepsilon}{\bar{v}_{i'}} \mathbf{e}_{i'} + \tfrac{\varepsilon}{\bar{v}_{i''}} \mathbf{e}_{i''} \succ \mathbf{y} \tag{26}$$

whenever $0 < \varepsilon \le (y_{i'} - y_{i''}) \min\{\bar{v}_{i'}, \bar{v}_{i''}\}$ and $F_{\mathbf{y}^\varepsilon} \ne F_{\mathbf{y}}$.

Alternatively, the fair dominance can be expressed on the cumulative distribution functions. Having introduced the right-continuous cumulative distribution function one may further integrate the cdf (23) to get the second order cumulative distribution function $F_{\mathbf{y}}^{(2)}(\tau) = \int_{-\infty}^{\tau} F_{\mathbf{y}}(\xi) d\xi$ representing the mean shortfall to any real target τ: $F_{\mathbf{y}}^{(2)}(\tau) = \sum_{i \in I} \bar{v}_i (\tau - y_i)_+$ (thus expanding the definition of $\bar{\delta}_\tau(\mathbf{y})$ (21) on the weighted case). By the theory of convex conjugate functions, the pointwise comparison of the second order cumulative distribution functions provides an alternative characterization of the fair dominance relation [39]. Exactly, $\mathbf{y}'$ fairly dominates $\mathbf{y}''$, iff $F_{\mathbf{y}'}^{(2)}(\tau) \le F_{\mathbf{y}''}^{(2)}(\tau)$ for all τ where at least one strict inequality holds.

Finally, there are three alternative analytical characterizations of the relation of fair dominance:

(i) $F_{\mathbf{y}'}^{(-2)}(\beta) \ge F_{\mathbf{y}''}^{(-2)}(\beta)$ for all $\beta \in (0, 1]$;

(ii) $F_{\mathbf{y}'}^{(2)}(\tau) \le F_{\mathbf{y}''}^{(2)}(\tau)$ for all real τ;

(iii) $\sum_{i \in I} \bar{v}_i u(y_i') \ge \sum_{i \in I} \bar{v}_i u(y_i'')$ for any concave, increasing function u.

Note that according to condition (iii), the fair dominance is actually the so-called increasing convex order which is more commonly known as the second degree stochastic dominance (SSD) [40]. Condition (i) covers the ordered outcome approaches (17) while the condition (ii) generates the multiple targets approaches (22). Actually, classical results of majorization theory [40] relate the mean utility comparison of condition (iii) to the comparison of the weighted mean shortfalls. Indeed, maximization of a concave and increasing utility function u is equivalent to minimization of the weighted aggregation of $F_{\mathbf{y}}^2(\tau)$ for several τ with positive weights representing minus second derivatives of the utility function u at τ. Similarly, the weighted aggregation may be applied to condition (i) thus generalizing the fair OWA solution concept (20) to the Weighted OWA (WOWA) or general Choquet integrals [41]. The fair WOWA optimization may be quite effectively implemented as an LP extension of the original problem [42].

VI. Conclusion

Within the networking applications the lexicographic maximin approach (or the MMF) is the most widely used fairness concept. Since, this approach may lead to significant losses in the overall efficiency (throughput of the network), a variety of techniques enabling to generate fair and efficient solutions were proposed. We have demonstrated that these solution concept may be viewed as some specific approaches to models of the fair optimization with the preference structure that complies with both the efficiency (Pareto-optimality) and with the Pigou-Dalton principle of transfers. Two alternative multiple criteria models equivalent to fair optimization have been introduced thus allowing to generate a variety of fair and efficient resource allocation pattern by possible using of the reference point approaches.

Fair allocation of multiple types of resources or more generally vector fair optimization approaches taking into account multi-attribute outcomes are still under-explored. Recently proposed (vector) fairness measure [43] allocates resources according to the MMF on dominant resource shares. Köppen [44] have extended the Jain's fairness index [23] to multi-attribute case by means of a lexicographic maximin procedure. Nevertheless, extension of the fair dominance models and several fair optimization concepts still remains an open problem.

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